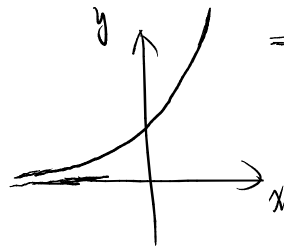
Limits at infinity

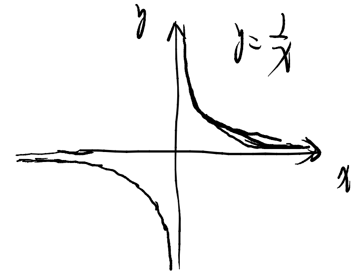
1. $\lim_{x\to+\infty} f(x) = L \iff$ f(x) can be arbitrarily close to L as x increases without bound

$\lim_{x\to-\infty} f(x) = L \iff$ f(x) can be arbitrarily close to as x decreases without bound.

2. $\lim_{x\to+\infty} \frac{1}{x} = 0$, $\lim_{x\to-\infty} \frac{1}{x} = 0$

$\lim_{x\to+\infty} e^x = +\infty$, $\lim_{x\to-\infty} e^x = 0$





3. computation.

Eg1. ① $\lim_{x\to+\infty} \frac{3x^2+2x-1}{2x^2-5} = \lim_{x\to+\infty} \frac{3+\overset{\to 0}{\frac{2}{x}}-\overset{\to 0}{\frac{1}{x^2}}}{2-\underset{\to 0}{\frac{5}{x^2}}} = \frac{3}{2}$

② $\lim_{x\to+\infty} \frac{x+5}{x^2-x+3} = \lim_{x\to+\infty} \frac{\frac{1}{x}+\frac{5}{x^2}}{1-\frac{1}{x}+\frac{3}{x^2}} = 0$

(method of compare the order)

③ $\lim_{x\to+\infty} \frac{6x^2-7}{x+3} = \lim_{x\to+\infty} \frac{6x-\frac{7}{x}}{1+\frac{3}{x}} = +\infty$

④ $\lim_{x\to+\infty} \frac{(2x+3)^{30}\cdot(3x-1)^{20}}{(5x+7)^{50}} = \lim_{x\to+\infty} \frac{(2x+3)^{30}\cdot(3x-1)^{20}/x^{50}}{(5x+7)^{50}/x^{50}}$

$= \lim_{x\to+\infty} \frac{\left(\frac{2x+3}{x}\right)^{30}\cdot\left(\frac{3x-1}{x}\right)^{20}}{\left(\frac{5x+7}{x}\right)^{50}} = \lim_{x\to+\infty} \frac{\left(2+\frac{3}{x}\right)^{30}\cdot\left(3-\frac{1}{x}\right)^{20}}{\left(5+\frac{7}{x}\right)^{50}} = \frac{2^{30}\cdot 3^{20}}{5^{50}}$

result: $f(x) = \frac{P(x)}{Q(x)}$

$\Rightarrow$ ① $\deg P(x) < \deg Q(x)$, $f(x) \to 0$ as $x \to \pm\infty$

② $\deg P(x) = \deg Q(x)$, $f(x) \to \frac{a_n}{b_n}$ a_n, b_n are leading coefficients of P(x) and Q(x) respectively

③ $\deg P(x) > \deg Q(x)$, $f(x) \to \pm\infty$

Eg2 (compare of order)

① $\lim \sqrt{5x^2-2} \;\; = \lim \frac{\sqrt{x^2\left(5-\frac{2}{x^2}\right)}}{} = \lim \frac{x\cdot\sqrt{5-\frac{2}{x^2}}}{}$

① $$\lim_{x\to+\infty}\frac{\sqrt{5x^2-2}}{x+3}=\lim_{x\to+\infty}\frac{\sqrt{x^2(5-\frac{2}{x^2})}}{x+3}=\lim_{x\to+\infty}\frac{x\cdot\sqrt{5-\frac{2}{x^2}}}{x+3}$$

$$=\lim_{x\to+\infty}\frac{\sqrt{5-\frac{2}{x^2}}}{1+\frac{3}{x}}=\sqrt{5}$$

② $$\lim_{x\to-\infty}\frac{\sqrt{5x^2-2}}{x+3}=\lim_{x\to-\infty}\frac{\sqrt{x^2(5-\frac{2}{x^2})}}{x+3}=\lim_{x\to-\infty}\frac{\sqrt{x^2}\cdot\sqrt{5-\frac{2}{x^2}}}{x+3}=\lim_{x\to-\infty}\frac{-x\cdot\sqrt{5-\frac{2}{x^2}}}{x+3}$$

$$=\lim_{x\to-\infty}\frac{-\sqrt{5-\frac{2}{x^2}}}{1+\frac{3}{x}}=-\sqrt{5}$$

$$\boxed{a^2-b^2=(a-b)(a+b)}$$

Eg3. (rationalize)

① $$\lim_{x\to+\infty}\left(\sqrt{x^2+x}-\sqrt{x^2-x}\right)=\lim_{x\to+\infty}\frac{(\sqrt{x^2+x}-\sqrt{x^2-x})(\sqrt{x^2+x}+\sqrt{x^2-x})}{\sqrt{x^2+x}+\sqrt{x^2-x}}=\lim_{x\to+\infty}\frac{(x^2+x)-(x^2-x)}{\sqrt{x^2+x}+\sqrt{x^2-x}}$$

$$=\lim_{x\to+\infty}\frac{2x}{\sqrt{x^2+x}+\sqrt{x^2-x}}=\lim_{x\to+\infty}\frac{2x}{x\cdot\sqrt{1+\frac{1}{x}}+x\sqrt{1-\frac{1}{x}}}=\lim_{x\to+\infty}\frac{2}{\sqrt{1+\frac{1}{x}}+\sqrt{1-\frac{1}{x}}}=1$$

② $$\lim_{x\to-\infty}\left(\sqrt{x^2+x}-\sqrt{x^2-x}\right)=\lim_{x\to-\infty}\frac{2x}{\sqrt{x^2+x}+\sqrt{x^2-x}}=\lim_{x\to-\infty}\frac{2x}{-x\sqrt{1+\frac{1}{x}}-x\sqrt{1-\frac{1}{x}}}=-1$$

Eg4: ① $$\lim_{x\to+\infty}\frac{e^x+e^{-x}}{e^x-e^{-x}}=\lim_{x\to+\infty}\frac{1+e^{-2x}}{1-e^{-2x}}$$

$$\boxed{e^{-x}/e^x=e^{-x}\cdot e^{-x}=e^{-2x}}$$

② $$\lim_{x\to-\infty}\frac{e^x+e^{-x}}{e^x-e^{-x}}=\lim_{x\to-\infty}\frac{e^{2x}+1}{e^{2x}-1}=-1$$