Implicit differentiation

1. $y = f(x)$: explicit function

$F(x,y)=0 \Rightarrow y = g(x)$

$F(x,y)=0$: Implicit function

Examples: 1. $xy + y + 1 = x \Rightarrow (x+1)y = x-1$

$\Rightarrow y = \frac{x-1}{x+1}$ (explicit function)

2. $e^{xy^2} + 2\sin(xy+y^2) = x^3 y$

3. $x^2+y^2=1$: $y = \pm\sqrt{1-x^2}$

2. Implicit differentiation: differentiate both sides of the equation w.r.t x, regard y as a function of x (regard y as intermediate variable in chain rule), the solve $\frac{dy}{dx}$ from the equation.

Example 1: If $x^2+y^2=1$, find $\frac{dy}{dx}$

$\frac{d}{dx}(y^2) = 2y \cdot \frac{dy}{dx}$

Solution: Take derivative both sides of the equation with respect to x,

$\Rightarrow 2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$

Example 2: $e^{xy^2} + 2\sin(xy+y^2) = x^3 y$

Solution: Differentiate both sides with respect to x,

$\Rightarrow e^{xy^2}\cdot\left(y^2 + x\cdot 2y\cdot\frac{dy}{dx}\right) + 2\sin(xy+y^2)\cdot\left(y + x\frac{dy}{dx} + 2y\frac{dy}{dx}\right) = 3x^2y + x^3\frac{dy}{dx}$

$\Rightarrow \left[2xye^{xy^2} + 2x\sin(xy+y^2) + 4y\sin(xy+y^2) - x^3\right]\frac{dy}{dx} = 3x^2y - y^2e^{xy^2} - 2y\sin(xy+y^2)$

$$\Rightarrow \frac{dy}{dx} = \frac{3x^2y - y^2e^{xy^2} - 2y\sin(xy+y^2)}{2xye^{xy^2} + 2x\sin(xy+y^2) + 4y\sin(xy+y^2) - x^3}$$

Example 3. If $x^3+y^3=3xy$, ① find $\frac{dy}{dx}$, ② find the equation of tangent line to the curve at $\left(\frac{3}{2}, \frac{3}{2}\right)$.

Solution: ① using implicit differentiation

$$\Rightarrow 3x^2+3y^2\cdot\frac{dy}{dx} = 3y+3x\frac{dy}{dx}$$

$$\Rightarrow (3y^2-3x)\frac{dy}{dx} = 3y-3x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{y-x^2}{y^2-x}$$

② $$\left.\frac{dy}{dx}\right|_{\left(\frac{3}{2}, \frac{3}{2}\right)} = \frac{\frac{3}{2}-\frac{9}{4}}{\frac{9}{4}-\frac{3}{2}} = -1$$

$\Rightarrow$ the equation of tangent line (slope-point form: $y-y_0 = m(x-x_0)$)

$y-\frac{3}{2} = -1\left(x-\frac{3}{2}\right)$ (slope-point)

$\Rightarrow$ $y = -x+3$ (slope-intercept)

$\Rightarrow$ $x+y=3$ (general)