Linear system

1. linear system (Gauss elimination)

Eg1.
$$\begin{cases} x+y+2z=9 & ① \\ 2x+4y-3z=1 & ② \\ 3x+6y-5z=0 & ③ \end{cases}$$

$$\begin{array}{c} \\ ②+①\times(-2) \\ ③+①\times(-3) \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right)$$

matrix (argument matrix)

solution. $② + ① \times (-2)$, $③ + ① \times (-3)$

$$\Rightarrow \begin{cases} x+y+2z=9 & ① \\ 2y-7z=-17 & ② \\ 3y-11z=-27 & ③ \end{cases}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{array}\right) ② \times (\tfrac{1}{2})$$

$② \times (\frac{1}{2})$

$$\Rightarrow \begin{cases} x+y+2z=9 & ① \\ y-\frac{7}{2}z=-\frac{17}{2} & ② \\ 3y-11z=-27 & ③ \end{cases}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 3 & -11 & -27 \end{array}\right) ③ + ② \times (-3)$$

$③ + ② \times (-3)$:

$$\Rightarrow \begin{cases} x+y+2z=9 & ① \\ y-\frac{7}{2}z=-\frac{17}{2} & ② \\ -\frac{1}{2}z=-\frac{3}{2} & ③ \end{cases}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{array}\right) ③ \times (-2)$$

row echelon form

$③ \times (-2)$

$$\Rightarrow \begin{cases} x+y+2z=9 & ① \\ y-\frac{7}{2}z=-\frac{17}{2} & ② \\ z=3 & ③ \end{cases}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{array}\right) \begin{array}{l} ① + ③ \times (-2) \\ ② + ③ \times \frac{7}{2} \end{array}$$

$② + ③ \times \frac{7}{2}$, $① + ③ \times (-2)$

$$\Rightarrow \begin{cases} x+y=3 & ① \\ y=2 & ② \\ z=3 & ③ \end{cases}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right) ① + ② \times (-1)$$

$① + ② \times (-1)$

$$\Rightarrow \begin{cases} x=1 \\ y=2 \end{cases}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \end{array}\right)$$

$$\Rightarrow \begin{cases} x = 1 \\ y = 2 \\ z = 3 \end{cases}$$

$$\sim \left(\begin{array}{ccc|c} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

reduced row echelon form

$$\Rightarrow \begin{cases} x_1 = 1 \\ x_2 = 2 \\ x_3 = 3 \end{cases}$$

2. row operation (row reduction) $\leftrightarrow$ Gauss Elimination
   ① interchange two rows;
   ② multiply one row by a non-zero number;
   ③ add one row by multiple of another row.

3. matrix: a table of m rows, n columns member. $A_{\underline{m \times n}}$

4. coefficient matrix, argument matrix

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_n \end{cases} \Leftrightarrow A\vec{x} = \vec{b}$$

coefficient matrix $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$

argument matrix: $(A, \vec{b}) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_n \end{array}\right)$ $\vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$, $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

5. row echelon form: ① any number below the first non-zero member of non-zero row are zeros
   ② the first non-zero member of non-zero row is at the right of the first non-zerom member of above row

reduced row echolon form: ① row echelon form
   ② leading member (first non-zero member) is 1.

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③ all members below and above leading member are 0

6. pivot number ⇔ leading member: first non-zero membe in row echelon form.

7. solving linear system: $(A, \vec{b}) \xrightarrow{\text{row operation}}$ reduced row echelon form

eg2: $\begin{cases} -x_1 + 2x_2 - 5x_3 = 2 \\ -2x_1 - 3x_2 + 4x_3 = 11 \\ 4x_1 - 7x_2 + 17x_3 = -7 \end{cases}$

solution. $(A, \vec{b}) = \left(\begin{array}{ccc|c} -1 & 2 & -5 & 2 \\ -2 & -3 & 4 & 11 \\ 4 & -7 & 17 & -7 \end{array}\right) \begin{array}{l} ②+①\times(-2) \\ ③+①\times 4 \end{array} \sim \left(\begin{array}{ccc|c} -1 & 2 & -5 & 2 \\ 0 & -7 & 14 & 7 \\ 0 & 1 & -3 & 1 \end{array}\right) ②\times(-\tfrac{1}{7})$

$\sim \left(\begin{array}{ccc|c} -1 & 2 & -5 & 2 \\ 0 & 1 & -2 & -1 \\ 0 & 1 & -3 & 1 \end{array}\right) ③+②\times(-1) \sim \left(\begin{array}{ccc|c} -1 & 2 & -5 & 2 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & -1 & 2 \end{array}\right) \begin{array}{l} ①\times(-1) \\ \\ ③\times(-1) \end{array}$

$\sim \left(\begin{array}{ccc|c} 1 & -2 & 5 & -2 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right) \begin{array}{l} ①+③\times(-5) \\ ②+③\times 2 \end{array} \sim \left(\begin{array}{ccc|c} 1 & -2 & 0 & 8 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -2 \end{array}\right) ①+②\times 2$

$\sim \underline{\left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -2 \end{array}\right)} \Rightarrow \underline{\begin{cases} x_1 = -2 \\ x_2 = -5 \\ x_3 = -2 \end{cases}}$